

ARITHMETICAL RANK OF STRINGS AND CYCLES

KYOUKO KIMURA¹ AND PAOLO MANTERO²

ABSTRACT. Let R be a polynomial ring over a field K . To a given squarefree monomial ideal $I \subset R$, one can associate a hypergraph $\mathcal{H}(I)$. In this article, we prove that the arithmetical rank of I is equal to the projective dimension of R/I when $\mathcal{H}(I)$ is a string or a cycle hypergraph.

INTRODUCTION

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and I a squarefree monomial ideal of R . The *arithmetical rank* of I , denoted by $\text{ara } I$, is defined as the minimum number u of elements $q_1, \dots, q_u \in R$ such that the equality

$$\sqrt{(q_1, \dots, q_u)} = \sqrt{I} (= I)$$

holds. When this is the case, one says that q_1, \dots, q_u generate I up to radical. Let $G(I)$ denote the minimal set of monomial generators of I and set $\mu(I) = \#G(I)$. Then $\text{ara } I \leq \mu(I)$ holds. On the other hand, Lyubeznik [15] proved that $\text{ara } I \geq \text{pd } R/I$, where $\text{pd } R/I$ denotes the projective dimension of R/I . Therefore we have

$$\text{height } I \leq \text{pd } R/I \leq \text{ara } I \leq \mu(I).$$

From the above inequalities, it is natural to ask when $\text{ara } I = \text{pd } R/I$ holds. Many authors including [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 16, 17, 18, 19] investigated this problem. In particular, in [10, 11] (see also [7]), Terai, Yoshida and the first author attacked the problem for ideals I with $\mu(I) - \text{height } I \leq 2$. Their idea is to classify these squarefree monomial ideals using hypergraphs (this classification is also used in [12]). The association of a hypergraph to a squarefree monomial ideal I of R with $G(I) = \{m_1, \dots, m_\mu\}$ is defined by setting

$$\mathcal{H}(I) := \{\{j \in [\mu] : x_i \mid m_j\} : i = 1, \dots, n\}.$$

$\mathcal{H}(I)$ is indeed a (separated) hypergraph on the vertex set $[\mu] := \{1, 2, \dots, \mu\}$. On the other hand, given a separated hypergraph \mathcal{H} , one can construct a squarefree monomial ideal I with $\mathcal{H}(I) = \mathcal{H}$; see Section 1 for more details.

Date: July 22, 2014.

AMS 2010 Mathematics Subject Classification: Primary 13F55; Secondary 13A15.

Keywords: arithmetical rank; projective dimension; squarefree monomial ideals; hypergraphs; free resolutions.

¹ K. Kimura was partially supported by JSPS Grant-in-Aid for Young Scientists (B) 24740008.

² P. Mantero was partially supported by an AMS-Simons Travel Grant.

We focus on the squarefree monomial ideals I such that $\mathcal{H}(I)$ is a string or a cycle. For these ideals, Lin and the second author [14] found an explicit formula expressing the projective dimension of R/I in terms of purely combinatorial invariants of the hypergraph $\mathcal{H}(I)$, namely

$$\mathrm{pd}(R/I) = \mu(I) - b(\mathcal{H}(I)) + M(\mathcal{H}(I)).$$

See the discussion before Theorem 2.3 for the definition of $b(\mathcal{H}(I))$ and $M(\mathcal{H}(I))$.

In the present work we study the arithmetical rank of these ideals I . We prove that $\mathrm{pd} R/I$ elements can be chosen so that they generate I up to radical, and have “small” monomial support. To be more precise, let us recall that the *binomial arithmetical rank* of I , denoted by $\mathrm{biara} I$, is the minimum number u of binomials or monomials $q_1, \dots, q_u \in R$ which generate I up to radical. Here we also define the *trinomial arithmetical rank* of I as the minimum number u of trinomials, binomials or monomials $q_1, \dots, q_u \in R$ which generate I up to radical. We denote it by $\mathrm{triara} I$. Clearly one has $\mathrm{ara} I \leq \mathrm{triara} I \leq \mathrm{biara} I$. Our main result is the following theorem.

Theorem 0.1. *Let I be a squarefree monomial ideal of R .*

- (1) *Assume that $\mathcal{H}(I)$ is a string hypergraph. Then $\mathrm{ara} I = \mathrm{biara} I = \mathrm{pd} R/I$.*
- (2) *Assume that $\mathcal{H}(I)$ is a cycle hypergraph. Then $\mathrm{ara} I = \mathrm{triara} I = \mathrm{pd} R/I$.*

In particular, the arithmetical rank of these ideals is independent of the characteristic of the field K . Crucial ingredients of our proof of Theorem 0.1 are a lemma by Schmitt and Vogel ([18], Lemma 3.2) and the above formula for the projective dimension (Theorem 2.3).

Now we explain the organization of this article. In Section 1, we recall the definition of the (separated) hypergraph associated to a squarefree monomial ideal, first introduced in [10]. In Section 2, we recall a few results by Lin and the second author [14] that will be employed in the subsequent sections. Then, in Sections 3 and 4, we prove Theorem 0.1 (1) and (2), respectively.

1. HYPERGRAPHS

In this section, we recall the construction of a separated hypergraph associated to any squarefree monomial ideal. The construction was introduced in [10]; see also [7, 11, 12, 14].

Set $V = [\mu]$. A collection $\mathcal{H} \subset 2^V$ is called a *hypergraph* on the vertex set V if $V = \bigcup_{F \in \mathcal{H}} F$. An element $F \in \mathcal{H}$ is called a *face* of \mathcal{H} . A vertex $j \in V$ is called *closed* (resp. *open*) if $\{j\} \in \mathcal{H}$ (resp. $\{j\} \notin \mathcal{H}$). A hypergraph is called *saturated* if $\{j\} \in \mathcal{H}$ for all $j \in V$. Let $i, j \in V$ be two vertices of \mathcal{H} . We say that i is a *neighbor* of j if there exists a face $F \in \mathcal{H}$ containing both i and j .

A hypergraph \mathcal{H} on V is said to be *separated* if for all vertices $i, j \in V$ ($i \neq j$), there exist faces $F, G \in \mathcal{H}$ such that $i \in F \setminus G$ and $j \in G \setminus F$. Let I be a squarefree monomial

ideal of $R = K[x_1, \dots, x_n]$ with $G(I) = \{m_1, \dots, m_\mu\}$. The hypergraph associated to I is defined as

$$\mathcal{H}(I) := \{\{j \in [\mu] : x_i \mid m_j\} : i = 1, \dots, n\},$$

which is a separated hypergraph on $[\mu]$.

Conversely, let \mathcal{H} be a separated hypergraph on $[\mu]$. Then we can construct a squarefree monomial ideal I with $\mathcal{H}(I) = \mathcal{H}$ in a polynomial ring with enough variables as follows: for each $F \in \mathcal{H}$, take a squarefree monomial m_F such that m_F and m_G are coprime if $F \neq G$. For each $j \in [\mu]$, set $m_j = \prod_{F \in \mathcal{H}, j \in F} m_F$. Then $I = (m_1, \dots, m_\mu)$ is a squarefree monomial ideal with $\mathcal{H}(I) = \mathcal{H}$. This construction implies that there are many ideals I (in various polynomial rings) with $\mathcal{H}(I) = \mathcal{H}$. We set $I(\mathcal{H})$ to be the ideal obtained from the above construction by setting each m_F to be a variable x_F in a polynomial ring $R(\mathcal{H}) := K[x_F : F \in \mathcal{H}]$.

The above correspondence between squarefree monomial ideals and separated hypergraphs yields the classification of squarefree monomial ideals mentioned in the introduction. The following proposition shows the usefulness of this association for our purpose.

Proposition 1.1 ([14, Corollary 2.4], [7, Proposition 3.2]). *Let I_1, I_2 be squarefree monomial ideals with $\mathcal{H}(I_1) = \mathcal{H}(I_2)$. Then $\text{pd } I_1 = \text{pd } I_2$ and $\text{ara } I_1 = \text{ara } I_2$ hold.*

Let I be a squarefree monomial ideal of R . Set $\mathcal{H} = \mathcal{H}(I)$. By Proposition 1.1, the following notation is well-defined: $\text{pd}(\mathcal{H}) := \text{pd } R/I$, $\text{ara}(\mathcal{H}) := \text{ara}(I)$. We call $\text{pd}(\mathcal{H})$ (resp. $\text{ara}(\mathcal{H})$) the projective dimension (resp. arithmetical rank) of \mathcal{H} . We will compute $\text{pd}(\mathcal{H})$, $\text{ara}(\mathcal{H})$ by computing $\text{pd } R(\mathcal{H})/I(\mathcal{H})$, $\text{ara } I(\mathcal{H})$, respectively.

Remark 1.2. *The statement of Proposition 1.1 remains true if we replace the arithmetical rank by the binomial or the trinomial arithmetical rank. Hence, we use the similar notations $\text{biara}(\mathcal{H})$, $\text{triara}(\mathcal{H})$.*

2. PROJECTIVE DIMENSIONS OF A STRING HYPERGRAPH AND A CYCLE HYPERGRAPH

In this section, we collect results about the projective dimensions of a string hypergraph and a cycle hypergraph. These results are proved by Lin and the second author in [14].

We first recall the definitions of a string hypergraph and a cycle hypergraph.

Definition 2.1 ([14, Definition 2.13]). *Fix $\mu \geq 2$. A hypergraph \mathcal{H} on $V = [\mu]$ is a string if $\{j, j+1\} \in \mathcal{H}$ for all $j = 1, \dots, \mu-1$ and the only other possible faces of \mathcal{H} are of the form $\{j\}$, for some $j \in V$.*

For a string hypergraph \mathcal{H} on $[\mu]$, we call the vertices 1 and μ the endpoints of \mathcal{H} . Note that if \mathcal{H} is separated, then both endpoints are closed vertices.

Definition 2.2 ([14, Definition 4.1]). *Fix $\mu \geq 3$. A hypergraph \mathcal{H} on $V = [\mu]$ is a μ -cycle if \mathcal{H} can be written as $\mathcal{H} = \tilde{\mathcal{H}} \cup \{\{\mu, 1\}\}$ where $\tilde{\mathcal{H}}$ is a string hypergraph on $[\mu]$.*

To introduce the explicit formula for the projective dimension of a string hypergraph and a cycle hypergraph in terms of invariants of the hypergraph we need some more definitions.

A hypergraph \mathcal{H} on $[\mu]$ is called a *string of opens* if \mathcal{H} is a string hypergraph with $\mu \geq 3$ whose only closed vertices are its endpoint.

First we assume that \mathcal{H} is a string hypergraph. We set $s = s(\mathcal{H})$ to be the number of strings of opens inside \mathcal{H} . We number the strings of opens in \mathcal{H} from one endpoint to another and set $n_i(\mathcal{H})$ to be the number of open vertices in the i -th string of opens. We say that \mathcal{H} is a *2-special configuration* if $s \geq 2$, \mathcal{H} does not contain two adjacent closed vertices, $n_1 \equiv n_s \equiv 1 \pmod{3}$, and $n_i \equiv 2 \pmod{3}$ for $i = 2, \dots, s-1$. Two 2-special configurations contained in \mathcal{H} are said to be *disjoint* if they do not have a common open vertex. The *modularity* of \mathcal{H} , denoted by $M(\mathcal{H})$, is the maximum number of pairwise disjoint 2-special configurations contained in \mathcal{H} .

Next we assume that \mathcal{H} is a cycle hypergraph. If \mathcal{H} contains at least two closed vertices, we define $s = s(\mathcal{H})$ and $n_1(\mathcal{H}), \dots, n_s(\mathcal{H})$ analogously to the case of a string hypergraph. If \mathcal{H} contains at most one closed vertex, we set $s = s(\mathcal{H}) = 1$ and $n_1(\mathcal{H}) = \mu(\mathcal{H}) - 1$. In either case, the definition of a 2-special configuration \mathcal{S} in \mathcal{H} is the same as in the case of a string hypergraph, except for allowing that the two extremal vertices of \mathcal{S} coincide. The modularity $M(\mathcal{H})$ is defined in the same way as in the case of a string hypergraph.

Let \mathcal{H} be a string hypergraph or a cycle hypergraph. Set

$$b(\mathcal{H}) = s(\mathcal{H}) + \sum_{i=1}^{s(\mathcal{H})} \left\lfloor \frac{n_i(\mathcal{H}) - 1}{3} \right\rfloor.$$

Theorem 2.3 (Lin and Mantero [14, Theorems 3.4 and 4.3]). *Let \mathcal{H} be a string hypergraph or a cycle hypergraph. Then*

$$\text{pd}(\mathcal{H}) = \mu(\mathcal{H}) - b(\mathcal{H}) + M(\mathcal{H}).$$

We also collect some inductive results about the projective dimension.

Let I be a squarefree monomial ideal with $G(I) = \{m_1, \dots, m_\mu\}$. Then we set $I_i := (m_{i+1}, \dots, m_\mu)$ and $\mathcal{H}_i := \mathcal{H}(I_i)$. Also we set $J_1 := I_1 : m_1$ and $\mathcal{Q}_1 := \mathcal{H}(J_1)$.

Lemma 2.4 ([14, Lemmas 2.6 and 2.11]). *Let \mathcal{H} be a hypergraph on $[\mu]$ with $\mu \geq 2$. Assume that $\{1\} \in \mathcal{H}$. Then $\text{pd}(\mathcal{H}) = \max\{\text{pd}(\mathcal{H}_1), \text{pd}(\mathcal{Q}_1) + 1\}$. Moreover, if all the neighbors of 1 are closed vertices, then $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_1) + 1$.*

Finally, for a string hypergraph \mathcal{H} , we will use the following results that allow us to compare $\text{pd}(\mathcal{H})$ with the projective dimension of a smaller string hypergraph.

Lemma 2.5 ([14, Lemma 2.14 (ii)]). *Let \mathcal{H} be a string hypergraph on $[\mu]$ with $\mu \geq 3$. Then $\text{pd}(\mathcal{H}) \leq \text{pd}(\mathcal{H}_2) + 2$.*

Lemma 2.6 ([14, Proposition 2.15]). *Let \mathcal{H} be a string hypergraph on $[\mu]$ with $\mu \geq 4$. Assume $\{2\} \notin \mathcal{H}$. Then $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_3) + 2$.*

3. STRINGS

In this section, we consider string hypergraphs. The goal of this section is to prove the following result.

Theorem 3.1. *Let \mathcal{H} be a string hypergraph. Then $\text{ara}(\mathcal{H}) = \text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Before proving the theorem, we introduce a useful lemma by Schmitt and Vogel [18].

Lemma 3.2 ([18, Lemma, p. 249]). *Let R be a commutative ring and P a finite subset of R . Let P_0, P_1, \dots, P_u be subsets of P satisfying the following 3 conditions:*

$$(SV1) \quad \bigcup_{\ell=0}^u P_\ell = P.$$

$$(SV2) \quad \#P_0 = 1.$$

(SV3) *For any integer $\ell > 0$ and elements $p, p'' \in P_\ell$ with $p \neq p''$, there exist an integer $\ell' < \ell$ and an element $p' \in P_{\ell'}$ such that $pp'' \in (p')$.*

Let I be an ideal of R generated by P and set

$$q_\ell = \sum_{p \in P_\ell} p, \quad \ell = 0, 1, \dots, u.$$

Then q_0, q_1, \dots, q_u generate I up to radical.

We first see the case where the number of vertices is less than or equal to 3.

Lemma 3.3. *Let \mathcal{H} be a string hypergraph on $[\mu]$. If $\mu \leq 3$, then $\text{ara}(\mathcal{H}) = \text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. If \mathcal{H} is saturated, then $\text{pd}(\mathcal{H}) = \mu$ and there is nothing to prove. The remaining case is that $\mu = 3$ and the vertex 2 of \mathcal{H} is open. Then $I(\mathcal{H}) = (y_1x_1, x_1x_2, y_3x_2)$. In this case $\text{pd}(\mathcal{H}) = 2$. By Lemma 3.2, we have $x_1x_2, y_1x_1 + y_3x_2$ generate $I(\mathcal{H})$ up to radical. \square

Next we assume $\mu \geq 4$. We divide the proof into two cases, depending on whether the vertex 2 is closed or open.

Lemma 3.4. *Let \mathcal{H} be a string hypergraph on $[\mu]$. Assume the neighbor 2 of the endpoint 1 of \mathcal{H} is closed. If $\text{biara}(\mathcal{H}_1) = \text{pd}(\mathcal{H}_1)$, then $\text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. We first note that $\text{biara}(\mathcal{H}) \leq \text{biara}(\mathcal{H}_1) + 1$ since $I(\mathcal{H})$ has one more generator than $I(\mathcal{H}_1)$. We then have the chain of inequalities

$$\text{biara}(\mathcal{H}) \leq \text{biara}(\mathcal{H}_1) + 1 = \text{pd}(\mathcal{H}_1) + 1 = \text{pd}(\mathcal{H}) \leq \text{biara}(\mathcal{H}),$$

where the last equality follows by Lemma 2.4. Therefore, $\text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$. \square

Lemma 3.5. *Let \mathcal{H} be a string hypergraph on $[\mu]$ with $\mu \geq 4$. Assume the neighbor 2 of the endpoint 1 of \mathcal{H} is open. If $\text{biara}(\mathcal{H}_3) = \text{pd}(\mathcal{H}_3)$, then $\text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. Write $I(\mathcal{H}) = I_3 + I'$ where $I_3 = I(\mathcal{H}_3) = (m_4, \dots, m_\mu)$ and $I' = (m_1, m_2, m_3)$. Note that $\mathcal{H}(I')$ is a string hypergraph on the vertex set $[3]$. Since the vertex 2 of $\mathcal{H}(I')$ is open, we have $\text{biara } I' = 2$ by Lemma 3.3. We then have

$$\text{biara}(\mathcal{H}) = \text{biara}(I_3 + I') \leq \text{biara}(I_3) + \text{biara}(I') = \text{biara}(I_3) + 2 = \text{pd}(\mathcal{H}_3) + 2.$$

Since $\text{pd}(\mathcal{H}_3) + 2 = \text{pd}(\mathcal{H})$ by Lemma 2.6, and $\text{pd}(\mathcal{H}) \leq \text{biara}(\mathcal{H})$ always holds, we have $\text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$. \square

We can now prove Theorem 3.1.

Proof of Theorem 3.1. We prove it by induction on the number μ of vertices of \mathcal{H} .

If $\mu \leq 3$, then the statement follows by Lemma 3.3. We may then assume $\mu \geq 4$ and the statement is proved for string hypergraphs with less than μ vertices. Then both $\text{biara}(\mathcal{H}_1) = \text{pd}(\mathcal{H}_1)$ and $\text{biara}(\mathcal{H}_3) = \text{pd}(\mathcal{H}_3)$ hold, and the assertion follows from Lemmas 3.4 and 3.5. \square

4. CYCLES

In this section, we consider cycle hypergraphs. The goal of this section is to prove the following result.

Theorem 4.1. *Let \mathcal{H} be a cycle hypergraph. Then $\text{ara}(\mathcal{H}) = \text{triara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

We first consider the case where \mathcal{H} contains at most 1 closed vertex.

Lemma 4.2. *Let \mathcal{H} be a cycle hypergraph. If \mathcal{H} contains at most 1 closed vertex, then $\text{triara}(\mathcal{H}) = \text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

If \mathcal{H} does not contain any closed vertex, then $I(\mathcal{H})$ is also the edge ideal of a cycle. In [2, Propositions 2.2, 2.3 and 2.4], Barile et al. constructed binomials and monomials which generate this ideal up to radical. Below we show that the same construction with minor modifications works also for \mathcal{H} which contains precisely one closed vertex.

Proof of Lemma 4.2. Let \mathcal{H} be a μ -cycle. By assumption, we may assume that the monomial generators of $I(\mathcal{H})$ are following forms:

$$yx_1x_\mu, x_1x_2, x_2x_3, \dots, x_{\mu-1}x_\mu,$$

where x_1, x_2, \dots, x_μ are pairwise distinct variables and y is either a variable which is different from x_1, x_2, \dots, x_μ or $y = 1$. By Theorem 2.3, we have

$$\text{pd}(\mathcal{H}) = \mu - \left(1 + \left\lfloor \frac{\mu-2}{3} \right\rfloor\right).$$

We distinguish three cases.

Case 1: $\mu = 3m$ ($m \geq 1$).

In this case, $\text{pd}(\mathcal{H}) = 2m$. Consider the following $2m$ elements:

$$\begin{cases} q_0 = x_1x_2, \\ q_1 = yx_1x_\mu + x_2x_3, \\ \begin{cases} q_{2i} = x_{3i+1}x_{3i+2}, \\ q_{2i+1} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}, \end{cases} & i = 1, 2, \dots, m-1. \end{cases}$$

Lemma 3.2 (see also [2, Proposition 2.2]) yields that $q_0, q_1, \dots, q_{2m-1}$ generate $I(\mathcal{H})$ up to radical.

Case 2: $\mu = 3m + 1$ ($m \geq 1$).

In this case, $\text{pd}(\mathcal{H}) = 2m + 1$. Consider the following $2m$ elements:

$$\begin{cases} q_{2i} = x_{3i+2}x_{3i+3}, \\ q_{2i+1} = x_{3i+1}x_{3i+2} + x_{3i+3}x_{3i+4}, \end{cases} \quad i = 0, 1, 2, \dots, m-1.$$

Set $q_{2m} = yx_1x_{3m+1}$.

Lemma 3.2 (see also [2, Proposition 2.3]) now yields that q_0, q_1, \dots, q_{2m} generate $I(\mathcal{H})$ up to radical.

Case 3: $\mu = 3m + 2$ ($m \geq 1$).

In this case, $\text{pd}(\mathcal{H}) = 2m + 1$. Consider the following $2m$ elements:

$$\begin{cases} q_0 = x_1x_2, \\ q_1 = x_2x_3 + x_4x_5, \\ \begin{cases} q_{2i} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}, \\ q_{2i+1} = x_{3i+2}x_{3i+3} + x_{3i+4}x_{3i+5}, \end{cases} & i = 1, 2, \dots, m-1. \end{cases}$$

Set $q_{2m} = yx_1x_{3m+2} + x_{3m}x_{3m+1}$ (see also [2, Proposition 2.4]).

Set $J = (q_0, q_1, \dots, q_{2m})$. We claim $\sqrt{J} = I(\mathcal{H})$. It is clear that $J \subset I(\mathcal{H})$. Thus we prove $\sqrt{J} \supset I(\mathcal{H})$.

We first prove $x_1I(\mathcal{H}) \subset \sqrt{J}$. Since one has $q_0, q_1 \in J$, then $x_1 \cdot x_1x_2, x_1x_2x_3, x_1x_4x_5 \in \sqrt{J}$. We claim that

$$(4.1) \quad x_1x_{3i}x_{3i+1}, x_1x_{3i+2}x_{3i+3}, x_1x_{3i+4}x_{3i+5} \in \sqrt{J}, \quad i = 1, 2, \dots, m-1.$$

We prove this by induction on i .

For the case $i = 1$, we need to prove that $x_1x_3x_4, x_1x_5x_6, x_1x_7x_8 \in \sqrt{J}$. Since $x_1q_2 = x_1x_3x_4 + x_1x_5x_6 \in J$ and $x_1x_4x_5 \in \sqrt{J}$, Lemma 3.2 yields $x_1x_3x_4, x_1x_5x_6 \in \sqrt{J}$. Then, since $x_1q_3 = x_1x_5x_6 + x_1x_7x_8 \in J$ and $x_1x_5x_6 \in \sqrt{J}$, we also have $x_1x_7x_8 \in \sqrt{J}$.

Assume that (4.1) is true for $i-1$. Then since $x_1q_{2i} = x_1x_{3i}x_{3i+1} + x_1x_{3i+2}x_{3i+3} \in J$ and $x_1x_{3i+1}x_{3i+2} = x_1x_{3(i-1)+4}x_{3(i-1)+5} \in \sqrt{J}$, Lemma 3.2 yields $x_1x_{3i}x_{3i+1}, x_1x_{3i+2}x_{3i+3} \in \sqrt{J}$. Then $x_1q_{2i+1} = x_1x_{3i+2}x_{3i+3} + x_1x_{3i+4}x_{3i+5} \in J$ and $x_1x_{3i+2}x_{3i+3} \in \sqrt{J}$, hence we have $x_1x_{3i+4}x_{3i+5} \in \sqrt{J}$, as required.

Therefore (4.1) holds true for all i . Moreover, $q_{2m} = yx_1x_{3m+2} + x_3x_{3m+1} \in J$ and $x_1x_{3m+1}x_{3m+2} = x_1x_{3(m-1)+4}x_{3(m-1)+5} \in \sqrt{J}$. These two facts imply $x_1 \cdot yx_1x_{3m+2}, x_1x_{3m}x_{3m+1} \in \sqrt{J}$.

Hence we have $x_1I(\mathcal{H}) \subset \sqrt{J}$.

Next we prove $I(\mathcal{H}) \subset \sqrt{J}$. Since $x_1I(\mathcal{H}) \subset \sqrt{J}$, we have $yx_1^2x_{3m+2} \in \sqrt{J}$, whence $yx_1x_{3m+2} \in \sqrt{J}$. Since $q_{2m} \in J$, we also have $x_{3m}x_{3m+1} \in \sqrt{J}$. We now prove

$$(4.2) \quad x_{3i}x_{3i+1}, x_{3i+2}x_{3i+3}, x_{3i+4}x_{3i+5} \in \sqrt{J}, \quad i = 1, 2, \dots, m-1$$

by descending induction on i .

When $i = m-1$, since $x_{3m}x_{3m+1} \in \sqrt{J}$ and $q_{2(m-1)+1} = x_{3m-1}x_{3m} + x_{3m+1}x_{3m+2} \in J$, Lemma 3.2 gives $x_{3m-1}x_{3m}, x_{3m+1}x_{3m+2} \in \sqrt{J}$. Also, since $q_{2(m-1)} = x_{3m-3}x_{3m-2} + x_{3m-1}x_{3m} \in J$, we have $x_{3m-3}x_{3m-2} \in \sqrt{J}$.

Next, assume that (4.2) holds true for $i+1$. Since $q_{2i+1} = x_{3i+2}x_{3i+3} + x_{3i+4}x_{3i+5} \in J$ and $x_{3i+3}x_{3i+4} = x_{3(i+1)}x_{3(i+1)+1} \in \sqrt{J}$, then Lemma 3.2 yields $x_{3i+2}x_{3i+3}, x_{3i+4}x_{3i+5} \in \sqrt{J}$. Then $q_{2i} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3} \in J$, and so we have $x_{3i}x_{3i+1} \in \sqrt{J}$, as required.

Note that $x_1x_2 = q_0 \in J$. Also, since $q_1 = x_2x_3 + x_4x_5$ and $x_3x_4 \in \sqrt{J}$, then $x_4x_5 \in \sqrt{J}$. This completes the proof. \square

Next, we consider the case where the number of vertices is at most 4. In this case, we know $\text{ara}(\mathcal{H}) = \text{pd}(\mathcal{H})$ by [10]. We prove the following slightly more precise lemma.

Lemma 4.3. *Let \mathcal{H} be a cycle hypergraph on $[\mu]$ with $\mu \leq 4$, then $\text{triara}(\mathcal{H}) = \text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. We first assume that $\text{pd}(\mathcal{H}) = \mu$. In this case, we can choose μ monomial generators. Next we assume that $\text{pd}(\mathcal{H}) < \mu$. In this case, we can easily check that $\text{pd}(\mathcal{H}) = \mu - 1$.

When $\mu = 3$, then the 3 generators of $I(\mathcal{H})$ can be written as $x_1x_2, y_1x_1x_3, y_2x_2x_3$, where each y_i can possibly be 1. By Lemma 3.2, $x_1x_2, y_1x_1x_3 + y_2x_2x_3$ generate $I(\mathcal{H})$ up to radical.

When $\mu = 4$, then the 4 generators of $I(\mathcal{H})$ can be written as $x_1x_2, y_1x_1x_4, y_2x_2x_3, y_3x_3x_4$, where each y_i is possibly 1. Lemma 3.2 yields that the elements $x_1x_2, y_1x_1x_4 + y_2x_2x_3, y_3x_3x_4$ generate $I(\mathcal{H})$ up to radical. \square

Thus, we can assume that the number of vertices of a cycle hypergraph is at least 5.

Lemma 4.4. *Let \mathcal{H} be a cycle hypergraph on $[\mu]$ with $\mu \geq 5$. If \mathcal{H} contains two adjacent closed vertices, then $\text{triara}(\mathcal{H}) = \text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. Without loss of generality we may assume 1 and μ are two adjacent closed vertices.

We first assume that the vertex 2 is also closed. Then we have $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_1) + 1$, by Lemma 2.4. Since \mathcal{H}_1 is a string hypergraph, we have $\text{biara}(\mathcal{H}_1) = \text{pd}(\mathcal{H}_1)$, by Theorem 3.1. Now, the equality $\text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$ follows because the monomial m_1 corresponding to the vertex 1, together with elements which generate $I(\mathcal{H}_1)$ up to radical, generate $I(\mathcal{H})$ up to radical (i.e. if $\sqrt{I(\mathcal{H}_1)} = \sqrt{(a_1, \dots, a_r)}$, then $\sqrt{I(\mathcal{H})} = \sqrt{(m_1, a_1, \dots, a_r)}$).

We may then assume that the vertex 2 is open. Then the monomials corresponding to the vertices 1, 2, 3 can be written as $y_1x_1x_\mu, x_1x_2, y_3x_2x_3$, respectively, where y_3 is possibly 1. Note that \mathcal{Q}_1 is the disjoint union of \mathcal{H}_3 and a closed vertex. Thus, $\text{pd}(\mathcal{Q}_1) = \text{pd}(\mathcal{H}_3) + 1$. By Lemma 2.4, we have

$$\text{pd}(\mathcal{H}) = \max\{\text{pd}(\mathcal{H}_1), \text{pd}(\mathcal{Q}_1) + 1\} = \max\{\text{pd}(\mathcal{H}_1), \text{pd}(\mathcal{H}_3) + 2\}.$$

Since \mathcal{H}_1 is a string hypergraph, we have $\text{pd}(\mathcal{H}_1) \leq \text{pd}(\mathcal{H}_3) + 2$ by Lemma 2.5, and thus $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_3) + 2$. Also, since \mathcal{H}_3 is a string hypergraph, Theorem 3.1 shows that $\text{biara}(\mathcal{H}_3) = \text{pd}(\mathcal{H}_3)$.

Since the elements $x_1x_2, y_1x_1x_\mu + y_3x_2x_3$, together with elements which generate I_3 up to radical, generate $I(\mathcal{H})$ up to radical, we obtain $\text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$. \square

In order to prove the following lemma, we use Theorem 2.3.

Lemma 4.5. *Let \mathcal{H} be a cycle hypergraph. Suppose that there is a string of opens with n_0 open vertices, with $n_0 \equiv 0 \pmod{3}$ in \mathcal{H} . Then $\text{triara}(\mathcal{H}) = \text{biara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. By Lemma 4.2, we may assume that \mathcal{H} contains at least 2 closed vertices. Let \mathcal{S}_0 be the string of opens with n_0 open vertices, and let u_1, u_2, u_3 be three adjacent open vertices in \mathcal{S}_0 such that u_1 is adjacent to a closed vertex v . Let v' be the other neighbor of u_3 . We consider the ideal I'' with $G(I'') = G(I) \setminus \{u_1, u_2, u_3\}$. Then, $\mathcal{H}'' := \mathcal{H}(I'')$ is a string hypergraph whose endpoints are v and v' (i.e., \mathcal{H}'' is obtained by *deletion* of the vertices u_1, u_2 and u_3 from \mathcal{H} and changing v' to be closed if v' is open in \mathcal{H}). We claim that $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}'') + 2$. Then, since we know that $\text{biara}(\mathcal{H}'') = \text{ara}(\mathcal{H}'') = \text{pd}(\mathcal{H}'')$, we can conclude that $\text{biara}(\mathcal{H}) = \text{ara}(\mathcal{H}) = \text{pd}(\mathcal{H})$, because $\text{ara}(\mathcal{H}'')$ elements which generate I'' up to radical, together with u_2 and $u_1 + u_3$, generate I up to radical.

Hence, we only need to prove the equality $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}'') + 2$. We first note that $\mu(\mathcal{H}'') = \mu(\mathcal{H}) - 3$ and that v' is a closed vertex in \mathcal{H}'' (independently of whether it is closed or not in \mathcal{H}).

If v' is closed in \mathcal{H} , then $s(\mathcal{H}'') = s(\mathcal{H}) - 1$. Since $\lfloor (n_0 - 1)/3 \rfloor = 0$, we have $b(\mathcal{H}'') = b(\mathcal{H}) - 1$. Moreover, $M(\mathcal{H}'') = M(\mathcal{H})$, because \mathcal{S}_0 does not belong to any 2-special configuration in \mathcal{H} . Therefore, we have $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}'') + 2$, by Theorem 2.3.

If v' is open in \mathcal{H} , then $s(\mathcal{H}'') = s(\mathcal{H})$. Let n_0'' be the number of open vertices in the string of opens \mathcal{H}'' , one of whose endpoints is v' . Then, $n_0'' = n_0 - 4 \equiv 2 \pmod{3}$. Note that $\lfloor (n_0 - 1)/3 \rfloor = n_0/3 - 1$ and $\lfloor (n_0'' - 1)/3 \rfloor = n_0/3 - 2$. Thus, $b(\mathcal{H}'') = b(\mathcal{H}) - 1$. Moreover, we have $M(\mathcal{H}'') = M(\mathcal{H})$, because both strings of opens do not belong to any 2-special configuration. Therefore, by Theorem 2.3, we have $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}'') + 2$. \square

By Lemma 4.5, we may then assume that each string of opens in \mathcal{H} contains a number of open vertices that is either congruent to $2 \pmod{3}$ or $1 \pmod{3}$.

Lemma 4.6. *If we prove that $\text{ara}(\mathcal{H}) = \text{triara}(\mathcal{H}) = \text{pd}(\mathcal{H})$ for a cycle hypergraph \mathcal{H} whose strings of opens all have at most 2 open vertices, then Theorem 4.1 follows.*

Proof. Let \mathcal{H} be a μ -cycle. By Lemma 4.2, we may assume \mathcal{H} has at least two closed vertices. By Lemma 4.3, we may assume $\mu \geq 5$. Moreover, by Lemma 4.4, we may assume that there are no two adjacent closed vertices in \mathcal{H} .

Suppose that \mathcal{H} contains a string of opens \mathcal{S} with $n_0 \geq 3$ open vertices. By Lemma 4.5, we may assume that $n_0 \equiv 1, 2 \pmod{3}$.

We first assume that $n_0 \equiv 1 \pmod{3}$. Let v be an endpoint of \mathcal{S} , and let u_1, u_2, u_3, u_4 be adjacent open vertices following v . Let \mathcal{H}' be the cycle hypergraph obtained by turning u_2 into a closed vertex. We claim that $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}')$.

Indeed, by the change we made, the string of opens \mathcal{S} in \mathcal{H} is now divided into two strings of opens \mathcal{S}_1 and \mathcal{S}_2 (in \mathcal{H}'), with 1 and $n_0 - 2$ open vertices, respectively. It is easy to see that $\mu(\mathcal{H}') = \mu(\mathcal{H})$, $s(\mathcal{H}') = s(\mathcal{H}) + 1$. Also, since $\lfloor (n_0 - 1)/3 \rfloor = (n_0 - 1)/3$, $\lfloor (1 - 1)/3 \rfloor + \lfloor ((n_0 - 2) - 1)/3 \rfloor = (n_0 - 1)/3 - 1$, we have $b(\mathcal{H}') = b(\mathcal{H})$. Moreover, the modularity is also unchanged because the change does not affect to the number of 2-special configurations. Now, the equality $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}')$ follows from the formula of Theorem 2.3.

Next, assume that $n_0 \equiv 2 \pmod{3}$. Let v be an endpoint of \mathcal{S} and let u_1, u_2, u_3, u_4, u_5 be adjacent open vertices following v . Let \mathcal{H}' be the cycle hypergraph obtained by turning u_3 into a closed vertex. We claim that $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}')$.

By the change, the string of opens \mathcal{S} in \mathcal{H} is now divided into two strings of opens \mathcal{S}_1 and \mathcal{S}_2 (in \mathcal{H}'), with 2 and $n_0 - 3$ open vertices, respectively. It is easy to see that $\mu(\mathcal{H}') = \mu(\mathcal{H})$, $s(\mathcal{H}') = s(\mathcal{H}) + 1$. Since $\lfloor (n_0 - 1)/3 \rfloor = (n_0 - 2)/3$, $\lfloor (2 - 1)/3 \rfloor + \lfloor ((n_0 - 3) - 1)/3 \rfloor = (n_0 - 2)/3 - 1$, we have $b(\mathcal{H}') = b(\mathcal{H})$. Furthermore, the modularity is also unchanged because the change does not affect to the number of 2-special configurations. All the above together with the formula of Theorem 2.3 implies the equality $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}')$.

Moreover, in either case, if $\text{triara}(\mathcal{H}') = \text{pd}(\mathcal{H}')$ then also $\text{triara}(\mathcal{H}) = \text{pd}(\mathcal{H})$ holds, as it can be seen by substituting 1 for the variables corresponding to the vertices which we made become closed.

Then, this procedure produces a new hypergraph $\tilde{\mathcal{H}}$ (obtained by making selected open vertices of \mathcal{H} become closed) and all strings of opens in $\tilde{\mathcal{H}}$ have at most 2 two open vertices. Moreover, the above shows that if $\text{triara}(\tilde{\mathcal{H}}) = \text{pd}(\tilde{\mathcal{H}})$, then $\text{triara}(\mathcal{H}) = \text{pd}(\mathcal{H})$ also holds. The statement now follows. \square

By the above results, we may then assume that \mathcal{H} is a cycle not containing two consecutive closed vertices and whose strings of opens have at most 2 open vertices. Note that for such a graph, $b(\mathcal{H}) = s(\mathcal{H})$ holds.

Next, we prove the case where there are strings of opens with precisely 2 open vertices.

Lemma 4.7. *Assume that \mathcal{H} contains a closed-open-open-closed string \mathcal{S} , where the two closed vertices of \mathcal{S} are distinct. Let \mathcal{H}' be the cycle hypergraph obtained by removing the 2 open vertices of \mathcal{S} from \mathcal{H} and identifying the two closed vertices of \mathcal{S} .*

If $\text{triara}(\mathcal{H}') = \text{pd}(\mathcal{H}')$, then $\text{triara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.

Proof. Let \mathcal{H} be a cycle hypergraph on $[\mu]$. By Lemma 4.4, we may assume that there are no two adjacent closed vertices in \mathcal{H} , and by Lemma 4.6 all strings of opens have at most two open vertices. We first claim that $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}') + 2$.

It is easy to see that $\mu(\mathcal{H}) = \mu(\mathcal{H}') + 3$ and $s(\mathcal{H}) = s(\mathcal{H}') + 1$. Since the removed string of opens has 2 open vertices, the modularity is unchanged. Hence, the claim follows by the formula of Theorem 2.3. To prove the statement we show that $\text{triara}(\mathcal{H}) \leq \text{triara}(\mathcal{H}') + 2$.

Let $1, \mu, \mu - 1, \mu - 2$ be the vertices of the string \mathcal{S} . We set the monomials corresponding to these vertices to be

$$(4.3) \quad y_1 x_1 x_\mu, x_{\mu-1} x_\mu, x_{\mu-2} x_{\mu-1}, y_{\mu-2} x_{\mu-3} x_{\mu-2}.$$

We set

$$\begin{cases} g_0 = y_1 y_{\mu-2} x_1 x_{\mu-3} x_{\mu-2} x_\mu, \\ g_1 = y_1 x_1 x_\mu + x_{\mu-2} x_{\mu-1}, \\ g_2 = y_{\mu-2} x_{\mu-3} x_{\mu-2} + x_{\mu-1} x_\mu. \end{cases}$$

We claim that

$$y_1 x_1 x_\mu, x_{\mu-1} x_\mu, x_{\mu-2} x_{\mu-1}, y_{\mu-2} x_{\mu-3} x_{\mu-2} \in \sqrt{(g_0, g_1, g_2)}.$$

Indeed, since

$$x_{\mu-2} x_{\mu-1} \cdot x_{\mu-1} x_\mu = (g_1 - y_1 x_1 x_\mu)(g_2 - y_{\mu-2} x_{\mu-3} x_{\mu-2}) \in (g_0, g_1, g_2),$$

we have $x_{\mu-2} x_{\mu-1} x_\mu \in \sqrt{(g_0, g_1, g_2)}$. Then the claim follows by Lemma 3.2.

Let I_0 be the squarefree monomial ideal which is generated by all monomials in $G(I(\mathcal{H}))$ except for the 4 monomials in (4.3). Then $I(\mathcal{H}) = I_0 + (y_1 x_1 x_\mu, x_{\mu-1} x_\mu, x_{\mu-2} x_{\mu-1}, y_{\mu-2} x_{\mu-3} x_{\mu-2})$. Let I' be the squarefree monomial ideal defined as $I' = I_0 + (y_1 y_{\mu-2} x_1 x_{\mu-3} x_{\mu-2} x_\mu)$ and note that $\mathcal{H}(I') = \mathcal{H}'$. Since $g_0 = y_1 y_{\mu-2} x_1 x_{\mu-3} x_{\mu-2} x_\mu \in I'$ it follows that $\text{ara}(\mathcal{H}')$ elements which generate I' up to radical, together with g_1, g_2 generate $I(\mathcal{H})$ up to radical. \square

Therefore, we reduce to the case of cycle hypergraphs in which closed vertices and open vertices appear alternately.

Lemma 4.8. *Let \mathcal{H} be a cycle hypergraph in which closed vertices and open vertices appear alternately. Then we have $\text{ara}(\mathcal{H}) = \text{triara}(\mathcal{H}) = \text{pd}(\mathcal{H})$.*

Proof. We first note that the number μ of vertices of \mathcal{H} is even.

Case 1: $\mu = 4m$.

By Theorem 2.3 we have $\text{pd}(\mathcal{H}) = 3m$, because $s(\mathcal{H}) = 2m$ and $M(\mathcal{H}) = m$. We now divide the vertices in disjoint groups of 4 adjacent vertices. In other words, there exist

m strings of the shape *closed-open-closed-open* in \mathcal{H} . It suffices to show that the ideal associated to any such string is generated up to radical by 3 polynomials each of which has at most 3 terms. So, let m_1, m_2, m_3, m_4 be monomials corresponding to the 4 vertices of the string, we can write m_1, m_2, m_3, m_4 as $y_1x_1x_\mu, x_1x_2, y_3x_2x_3, x_3x_4$. By Lemma 3.2, the following 3 polynomials

$$x_1x_2, y_1x_1x_\mu + y_3x_2x_3, x_3x_4$$

generate (m_1, m_2, m_3, m_4) up to radical, whence the statement follows.

Case 2: $\mu = 4m + 2$ ($m \geq 1$).

In this case, we prove the statement by induction on m . First assume $m = 1$. Then $I(\mathcal{H})$ is generated by the following 6 monomials:

$$y_1x_1x_6, x_1x_2, y_3x_2x_3, x_3x_4, y_5x_4x_5, x_5x_6.$$

By Theorem 2.3, we have $\text{pd}(\mathcal{H}) = 4$, and by Lemma 3.2 the following 4 polynomials generate $I(\mathcal{H})$ up to radical:

$$\begin{cases} x_1x_2, \\ x_3x_4, \\ x_5x_6, \\ y_1x_1x_6 + y_3x_2x_3 + y_5x_4x_5. \end{cases}$$

Now we assume that $m \geq 2$. In this case, \mathcal{H} contains a 2-special configuration \mathcal{S} : *closed-open-closed-open-closed*. Let $1, \mu, \mu - 1, \mu - 2, \mu - 3$ be the vertices of the string \mathcal{S} . We set the monomials corresponding to these vertices to be

$$(4.4) \quad y_1x_1x_\mu, x_{\mu-1}x_\mu, y_{\mu-1}x_{\mu-2}x_{\mu-1}, x_{\mu-3}x_{\mu-2}, y_{\mu-3}x_{\mu-4}x_{\mu-3}.$$

Let \mathcal{H}' be the cycle hypergraph obtained by removing the 3 inner vertices $\mu, \mu - 1, \mu - 2$ of \mathcal{S} from \mathcal{H} and identifying the two endpoints 1 and $\mu - 3$ of \mathcal{S} . Then \mathcal{H}' is the cycle hypergraph with $\mu(\mathcal{H}') = 4(m - 1) + 2$ in which closed vertices and open vertices appear alternately. Note that Theorem 2.3 yields $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}') + 3$, because $\mu(\mathcal{H}) = \mu(\mathcal{H}') + 4$, $s(\mathcal{H}) = s(\mathcal{H}') + 2$, and $M(\mathcal{H}) = M(\mathcal{H}') + 1$.

Let I_0 be the squarefree monomial ideal which is generated by all monomials in $G(I(\mathcal{H}))$ except for the 5 monomials in (4.4). Then

$$I(\mathcal{H}) = I_0 + (y_1x_1x_\mu, x_{\mu-1}x_\mu, y_{\mu-1}x_{\mu-2}x_{\mu-1}, x_{\mu-3}x_{\mu-2}, y_{\mu-3}x_{\mu-4}x_{\mu-3}).$$

We set $I' = I_0 + (y_1y_{\mu-3}x_1x_{\mu-4}x_{\mu-3}x_\mu)$. Note that $\mathcal{H}(I') = \mathcal{H}'$ and $y_1y_{\mu-3}x_1x_{\mu-4}x_{\mu-3}x_\mu$ is the monomial corresponding to the vertex 1 of $\mathcal{H}(I')$. Since $y_1y_{\mu-3}x_1x_{\mu-4}x_{\mu-3}x_\mu \in I'$, the following 3 polynomials, together with $\text{ara}(\mathcal{H}')$ elements which generate I' up to radical,

generate $I(\mathcal{H})$ up to radical:

$$\begin{cases} x_{\mu-1}x_{\mu}, \\ x_{\mu-3}x_{\mu-2}, \\ y_1x_1x_{\mu} + y_{\mu-1}x_{\mu-2}x_{\mu-1} + y_{\mu-3}x_{\mu-4}x_{\mu-3}. \end{cases}$$

□

Acknowledgement. The authors thank the referee for reading our manuscript carefully.

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, SHIZUOKA UNIVERSITY, 836 OHYA,
SURUGA-KU, SHIZUOKA 422-8529, JAPAN

E-mail address: `skkimur@ipc.shizuoka.ac.jp`

UNIVERSITY OF CALIFORNIA, RIVERSIDE, DEPARTMENT OF MATHEMATICS, RIVERSIDE, CA 92521

E-mail address: `mantero@math.ucr.edu`

URL: `http://math.ucr.edu/~mantero/`